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A Discrete Korovkin Theorem

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In this paper we give a sufficient condition for the pointwise Korovkin property on $B(X)$, the space of bounded real valued functions on an arbitrary countable set $X = \{x_1, ..., x_j, ...\}$. Our theorem follows from its $L_p(X, \mu)$ analogue (and conversely); here $1 \leqslant p < \infty$ and μ is a positive finite measure on X such that $\mu(\{x_j\}) > 0$ for all j. © 1985 Academic Press, Inc.

P. P. Korovkin [5, pp. 39, 49] proved the following fundamental theorem, concerning the convergence of a sequence of positive linear operators to the identity operator.

THEOREM 1. *Let* $f_1, f_2, f_3 \in C([a, b])$, *where* $-\infty < a < b < \infty$. *A necessary and sufficient condition that for every sequence* L_n *of positive linear operators on* $C([a, b])$ *the relations*

$$L_n(f_i) \to f_i, \qquad i = 1, 2, 3, \text{ uniformly in } [a, b]$$

imply

$$L_n(f) \to f, \qquad \text{uniformly in } [a, b],$$

for every $f \in C[a, b]$, *is that* f_1, f_2, f_3 *is a Chebyshev system on* $[a, b]$.

The next theorem gives a sufficient condition for the above Korovkin characterization in a much more general setting and also motivates Theorem 3, the main result of this paper.

THEOREM 2 (V. Volkov; see [8, Theorem 1]). *Let Q be a compact metric space. Let a sequence of positive linear operators L_n: $C(Q) \to C(Q)$ be such that*

$$L_n(f_i) \to f_i, \qquad i = 1, ..., k, \text{ uniformly in } Q.$$

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In order that $L_n(f) \to f$, uniformly in Q, for all $f \in C(Q)$, it is enough to assume that for each $x_0 \in Q$ there are real constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i f_i(x) \geqslant 0 \qquad \text{for all } x \in Q$$

and (2.1)

$$\sum_{i=1}^{k} \beta_i f_i(x) > 0 \qquad \text{for all } x \in Q - \{x_0\}.$$

Šaškin (see [7]) has shown that the sufficient condition of Volkov is not necessary.

THEOREM 3. *Let $X = \{x_1, ..., x_j, ...\}$ be a countable set. Consider $B(X)$, the space of real valued bounded functions on X with the supremum norm $\|\cdot\|_\infty$ and a sequence of positive linear operators L_n: $B(X) \to B(X)$ such that $L_n(1, x_j) = 1$ for all j. Suppose that, for some $\{f_1, ..., f_k\} \subset B(X)$,*

$$\lim_{n \to \infty} L_n(f_i, x_j) = f_i(x_j) \qquad \text{for all } i \text{ and } j. \tag{3.1}$$

In order that $L_n(f, x_j) \to f(x_j)$ for all $f \in B(X)$ and all x_j, it is enough to assume for each j there are real constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i (f_i(x) - f_i(x_j)) \geqslant 0 \qquad \text{for all } x \in X$$

and (3.2)

$$\sum_{i=1}^{k} \beta_i (f_i(x) - f_i(x_j)) > 0 \qquad \text{for all } x \in X - \{x_j\}.$$

We shall use the following

LEMMA 4. *Let $X = \{x_1, ..., x_j, ...\}$ be a countable measurable space, $1 \leqslant p < \infty$, and μ a finite positive measure on X such that $\mu(\{x_j\}) > 0$ for all j. Let $B(X)$ be as above and $f, f_1, f_2, ... \in B(X)$, where all $\|f_n\|_\infty < c$, $c > 0$. Then $f_n \to f$ pointwise on X iff $f_n \to f$ in $L_p(X, \mu)$.*

Proof. ($\Rightarrow$) By the uniform boundedness of f_n and by $\mu(X) < \infty$, we obtain $|f_n|^p \leqslant c^p \in L_1(X, \mu)$. Since $f_n \to f$ pointwise, by a variation of the dominated convergence theorem (see [4], p. 180) we get $f_n \to f$ in the pth mean. Note $B(X) \subset L_p(X, \mu)$.

($\Leftarrow$) The L_p convergence implies weak convergence, the indicator function $I_{\{x_j\}} \in L_q(X, \mu)$ where $1/p + 1/q = 1$ and $\mu(\{x_j\}) > 0$. Hence the pointwise convergence. ▌

Next is an independent L_p result which will be used in the proof of Theorem 3.

PROPOSITION 5. *Let $X = \{x_1, ..., x_j, ...\}$ be a countable set. Let $w(x_j) > 0$ for all j and $\sum_{j=1}^{\infty} w(x_j) < \infty$. Let $B(X)$ be as above and L_n be a sequence of positive linear operators: $B(X) \to B(X)$ such that $L_n(1, x_j) = 1$ for all j. Suppose that, for some $f_1, f_2, ..., f_k \in B(X)$ and some p, $1 \leqslant p < \infty$,*

$$\lim_{n \to \infty} \left(\sum_{j=1}^{\infty} |L_n(f_i, x_j) - f_i(x_j)|^p \, w(x_j) \right) = 0, i = 1, 2, ..., k. \quad (5.1)$$

In order that $\sum_{j=1}^{\infty} |L_n(f, x_j) - f(x_j)|^p \, w(x_j) \to 0$ for all $f \in B(X)$ it is enough to assume: for each j there are real constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i(f_i(x) - f_i(x_j)) \geqslant 0 \qquad \textit{for all } x \in X$$

and (5.2)

$$\sum_{i=1}^{k} \beta_i(f_i(x) - f_i(x_j)) > 0 \qquad \textit{for all } x \in X - \{x_j\}.$$

Proof. The weight w gives rise to a positive finite measure μ on X with $\mu(\{x\}) > 0$ for all $x \in X$. Since $B(X) \subset L_p(X, \mu)$, (5.1) implies $\|L_n(f_i) - f_i\|_p \to 0$ for all i. If there exists $f \in B(X)$ such that $\|L_n(f) - f\|_p \nrightarrow 0$, then there are $x_j \in X$ and an $\varepsilon > 0$ so that

$$|L_n(f, x_j) - f(x_j)| > \varepsilon \qquad \text{for all } n \geqslant \text{some } n_0.$$

Because each positive linear functional $L_n(\cdot, x_j)$ on $B(X)$ is bounded, by a basic representation theorem, for each specific $j = j_0$ as above, there exists $g_{x_{j_0}, n} \in L_q(X, \mu)$ where $1/p + 1/q = 1$ such that

$$L_n(f, x_{j_0}) = \int_X f(x) \cdot g_{x_{j_0}, n}(x) \cdot \mu(dx) \qquad \text{for all } f \in B(X).$$

By $L_n(1, x_{j_0}) = 1$ and the positivity of $L_n(\cdot, x_{j_0})$ one obtains $\int_X g_{x_{j_0}, n}(x) \cdot \mu(dx) = 1$ and $g_{x_{j_0}, n}(x) \geqslant 0$ for all $x \in X$. Thus

$$\begin{aligned}\varepsilon &< |L_n(f, x_{j0}) - f(x_{j0})| \\ &= \left| \int_{X - \{x_{j0}\}} (f(x) - f(x_{j0})) \cdot g_{x_{j0},n}(x) \cdot \mu(dx) \right| \\ &\leq \| f - f(x_{j0}) \|_\infty \cdot \left(\int_{X - \{x_{j0}\}} g_{x_{j0},n}(x) \cdot \mu(dx) \right),\end{aligned}$$

so

$$1 \geq \int_{X - \{x_{j0}\}} g_{x_{j0},n}(x) \cdot \mu(dx) > \frac{\varepsilon}{\| f - f(x_{j0}) \|_\infty} =: \delta > 0, \qquad \text{for all } n \geq n_0.$$

There cannot be constants $0 < \alpha_0 < \delta$ and $\alpha_1, ..., \alpha_k$ with

$$\alpha_0 + \sum_{i=1}^{k} \alpha_i \cdot (f_i(x) - f_i(x_{j0})) \geq 0, \qquad \text{for all } x \in X,$$

and

$$\alpha_0 + \sum_{i=1}^{k} \alpha_i \cdot (f_i(x) - f_i(x_{j0})) \geq 1, \qquad \text{for all } x \in X - \{x_{j0}\},$$

since, otherwise, we would have

$$\alpha_0 \cdot g_{x_{j0},n}(x) + \sum_{i=1}^{k} \alpha_i \cdot (f_i(x) - f_i(x_{j0})) \cdot g_{x_{j0},n}(x) \geq g_{x_{j0},n}(x),$$

for all $x \in X - \{x_{j0}\}$ and therefore

$$\begin{aligned}&\alpha_0 \cdot \int_{X - \{x_{j0}\}} g_{x_{j0},n}(x) \cdot \mu(dx) \\ &\qquad + \sum_{i=1}^{k} \alpha_i \cdot \int_{X - \{x_{j0}\}} (f_i(x) - f_i(x_{j0})) \cdot g_{x_{j0},n}(x) \cdot \mu(dx) \\ &\geq \int_{X - \{x_{j0}\}} g_{x_{j0},n}(x) \cdot \mu(dx).\end{aligned}$$

(Note that $L_n(f_i, x_{j0}) = \int_X f_i(x) \cdot g_{x_{j0},n}(x) \cdot \mu(dx)$, $i = 1, ..., k$.) Consequently, since $L_n(f_i, x_{j0}) \to f_i(x_{j0})$, $i = 1, ..., k$, we would get

$$0 = \lim_{n \to \infty} \left(\sum_{i=1}^{k} \alpha_i \cdot (L_n(f_i, x_{j0}) - f_i(x_{j0})) \right) \geq \delta \cdot (1 - \alpha_0)$$

and hence $\alpha_0 \geqslant 1$, contradicting the above relations $\alpha_0 < \delta < 1$. Therefore there cannot be constants $\beta_1, ..., \beta_k$ such that

$$\sum_{i=1}^{k} \beta_i(f_i(x) - f_i(x_{j_0})) \geqslant 0 \qquad \text{for all } x \in X$$

and

$$\sum_{i=1}^{k} \beta_i(f_i(x) - f_i(x_{j_0})) > 0 \qquad \text{for all } x \in X - \{x_{j_0}\}. \quad \blacksquare$$

Proof of Theorem 3. Note $B(X) \subset L_p(X, \mu)$ for any p, $1 \leqslant p < \infty$, and any positive finite measure μ on X with each $\mu(\{x_j\}) > 0$. By Lemma 4, the pointwise convergence $L_n(f_i, x_j) \to f_i(x_j)$, $i = 1, ..., k$, for all j, is equivalent for such p and μ, to the convergence in the pth mean of $L_n(f_i)$ to f_i, $i = 1, ..., k$. Furthermore, this measure μ can serve as a weight function on X. Thus Proposition 5 implies our theorem. $\blacksquare$

To display the power of conditions (3.2), we show that they are satisfied by basic Chebyshev systems such as $\{1, x, x^2\}$ and $\{1, x, e^x\}$.

EXAMPLES. Let $X = \{x_1, ..., x_j, ...\}$ be a real countable set with all $x_j \leqslant \tau$.

(i) The set $\{1, x, x^2\}$ satisfies (3.2), namely: for arbitrary $\beta_3 > 0$ and $\beta_2 = -2 \cdot \beta_3 \cdot x_j$ we have

$$\begin{aligned} \beta_2 \cdot (x - x_j) + \beta_3 \cdot (x^2 - x_j^2) &> 0 \qquad \text{for all } x \neq x_j \\ &= 0 \qquad \text{for } \quad x = x_j. \end{aligned}$$

(ii) Similarly, the $\{1, x, e^x\}$ fulfills (3.2): for arbitrary $\beta_3 > 0$ and $\beta_2 = -\beta_3 \cdot e^{x_j}$ we have

$$\begin{aligned} \beta_2 \cdot (x - x_j) + \beta_3 \cdot (e^x - e^{x_j}) &> 0 \qquad \text{for all } x \neq x_j \\ &= 0 \qquad \text{for } \quad x = x_j. \end{aligned}$$

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